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ON MIXED FINITE ELEMENT METHODS

II.   THE LEAST SQUARES METHOD

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## ABSTRACT

Continuing earlier work of the authors, a theoretical framework for the least squares solution of first order elliptic systems is proposed, and optimal order error estimates for piecewise polynomial approximation spaces are derived.

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# I. Introduction

The previous paper [1] in this series dealt with finite element approximations based on the Kelvin Principle. The latter was shown to be dual in a suitable sense to the classical Dirichlet Principle, and in many respects the computational properties of these principles are complementary. For example, when the classical Dirichlet Principle is used for Poisson's equation $\Delta\phi = f$, one directly approximates the scalar valued function $\phi$ with say linear finite elements. The resulting approximation $\phi_h$ satisfies[*]

$$(1) \qquad \| \phi - \phi_h \|_0 \leq C h^2 \| \phi \|_2$$

while there is a less accurate approximation to the gradient, i.e.,

$$(2) \qquad \| \nabla\phi - \nabla\phi_h \|_0 \leq C h \| \nabla\phi \|_1 \quad .$$

Conversely, if the Kelvin Principle is used and if our Grid Decomposition Property [1] is satisfied then the approximation $\underline{u}_h$ to $\underline{u} = \nabla\phi$ satisfies

$$(3) \qquad \| \underline{u}_h - \underline{u} \|_0 \leq C h^2 \| \underline{u} \|_2$$

while

$$(4) \qquad \| \phi_h - \phi \|_0 \leq C h ( \| \phi \|_1 + \| \underline{u} \|_2 ) \quad .$$

As we shall show in this paper the least squares approach is in essence a mixture of the Dirichlet and Kelvin Principles. It is natural to use the same spaces for both $\phi$ and $\underline{u}$ in least squares, and if say linear elements are used, then least squares inherits the property from the Dirichlet Principle

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[*]Throughout this paper $H^r(\Omega)$ denotes the r-th order Sobolev space with norm $\| \bullet \|_r$ (see [2]).

that the error in the scalar $\phi$ satisfies (1). On the other hand, least
squares inherits from the Kelvin Principle the property that (3) holds
if the Grid Decomposition Property is valid.

Least squares methods appear to be particularly applicable for in-
definite problems such as the Helmholtz equation

$$\Delta\phi + \omega^2\phi = f$$

arising in acoustics and elsewhere [3], [4]. This is especially true
since least squares techniques lead to algebraic problems with sparse
positive definite matrices. The Kelvin and Dirichlet Principles, on
the other hand, typically give indefinite matrix problems.

## II. Formulation of Problem

To fix ideas we consider the following boundary value problem: given
a function f, we seek a suitably smooth $\phi_0$ satisfying

(1) $$\Delta\phi_0 + q\phi_0 = f \quad \text{in } \Omega$$

(2) $$\phi_0 = 0 \quad \text{on } \Gamma_D$$

(3) $$\frac{\partial\phi_0}{\partial\nu} = 0 \quad \text{on } \Gamma_N$$

or what is the same

(4) $$\text{div}(\underline{u}_0) + q\phi_0 = f \quad \text{in } \Omega$$

(5) $$\nabla\phi_0 - \underline{u}_0 = 0 \quad \text{in } \Omega$$

(6) $$\phi_0 = 0 \quad \text{on } \Gamma_D, \quad \underline{u}_0 \cdot \underline{\nu} = 0 \quad \text{on } \Gamma_N$$

Here, $\Omega$ is a bounded open region of $\mathbb{R}^n$ with boundary $\Gamma_D \cup \Gamma_N$ and $\underline{\nu}$ is the outward directed normal to the boundary. To be precise we assume

$$f \in L_2(\Omega)$$

and we seek solutions $\phi_0$, $\underline{u}_0$ to (4)-(6) in

$$S_0 = \{\psi \mid \psi \in H^1(\Omega), \; \psi = 0 \text{ on } \Gamma_D\}$$

and

$$\underline{V}_0 = \{\underline{v} \mid \underline{v} \in \underline{H}^1(\Omega), \; \underline{v} \cdot \underline{\nu} = 0 \text{ on } \Gamma_N\} \; .$$

The standard least squares method for (4)-(6) introduces finite dimensional subspaces

$$\underline{V}_0^h \subset \underline{V}_0 \quad \text{and} \quad S_0^\delta \subset S_0$$

and minimizes the residual in the following sense:

### Least Squares Variational Principle

Find a $\phi_\delta \in S_0^\delta$ and a $\underline{u}_h \in \underline{V}_0^h$ which minimizes

$$(7) \qquad \int_\Omega \{ |\nabla \psi^\delta - \underline{v}^h|^2 + |\text{div} \, \underline{v}^h + q\psi^\delta - f|^2 \}$$

over $\psi^\delta \in S_0^\delta$ and $\underline{v}^h \in \underline{V}_0^h$.

Taking the first variation gives

$$(8) \quad \int_\Omega (\nabla\phi_\delta - \underline{u}_h)(\nabla\psi^\delta - \underline{v}^h) + (\text{div}(\underline{u}_h) + q\phi_\delta)(\text{div}(\underline{v}^h) + q\psi^\delta) = \int_\Omega f(\text{div}(\underline{v}^h) + q\psi^\delta)$$

a relation which holds for all $\underline{v}^h \in \underline{V}_0^h$ and all $\psi^\delta \in S_0^\delta$. A fact that will be needed in the next section is that (8) remains valid when $\psi_\delta$ is replaced with $\psi_0$ and $\underline{u}_h$ replaced with $\underline{u}_0$; i.e.,

$$(9) \quad \int_\Omega (\nabla\phi_0 - \underline{u}_0)(\nabla\psi^\delta - \underline{v}^h) + (\text{div}(\underline{u}_0) + q\phi_u)(\text{div}(\underline{v}^h) + q\psi^\delta) = \int_\Omega f\{\text{div}(\underline{v}^h) + q\psi^\delta\}.$$

We shall assume throughout the standard [5]-[6] approximation properties for the finite dimensional spaces $\underline{V}_0^h$, $S_0^\delta$ in terms of the Sobolev norms $\| \cdot \|_r$ on $H^r(\Omega)$. In particular, we shall need the following:

## Approximation property

There are integers $k \geq 1$ and $\ell \geq 1$, parameters $0 < h < h_0$ and $0 < \delta < \delta_0$ and finite dimensional spaces

$$\{\underline{V}_0^h\} \; 0 < h < h_0 \qquad , \quad \{S^\delta\} \; 0 < \delta < \delta_0$$

for which the following holds: for all $\underline{u} \in H^k(\Omega) \cap \underline{V}_0$ and $\phi \in S_0 \cap H^\ell(\Omega)$ there are $\hat{\underline{u}}_h \in \underline{V}_0^h$ and $\hat{\phi}_\delta \in S_0^\delta$ such that

(10) $$\| \underline{u} - \hat{\underline{u}}_h \|_t \; \leq \; C_A \, h^{k-t} \| \underline{u} \|_k$$

(11) $$\| \phi - \hat{\phi}_\delta \|_t \; \leq \; C_A \, \delta^{\ell-t} \| \phi \|_\ell$$

for $t = 0$ and $t = 1$ where $C_A$ is a constant independent of $h$, $\delta$, $\underline{u}$ and $\phi$.

The remaining assumption used in the theory is the Grid Decomposition Property alluded to in the introduction and discussed in detail in [1]. A precise statement of this property is as follows:

## Grid Decomposition Property

For each $\underline{v}_h \in \underline{V}_0^h$ there exists $\underline{w}_h$ and $\underline{z}_h$ in $\underline{V}_0^h$ such that

$$\underline{v}_h = \underline{w}_h + \underline{z}_h$$

with $\operatorname{div} \underline{z}_h = 0$ and

$$\int_\Omega \underline{z}_h \cdot \underline{w}_h = 0$$

and such that

$$\| \underline{w}_h \|_0 \leq C_G \| \operatorname{div} \underline{v}_h \|_{-1}$$

for some positive constant $C_G$ independent of $h$ and $\underline{v}^h$.

In the next section, it will be proved that the $L_2$ error

(12) $$\varepsilon = \phi_0 - \phi_\delta$$

in the least squares approximation is the best possible. This result generalizes the work of Jesperson [7] who proved it for Laplace's equation. Jesperson was unable to obtain optimal estimates for

(13) $$\underline{e} = \underline{u}_0 - \underline{u}_h$$

and as we shall show in the third paper in this series through numerical experiments there is, in general, a loss of accuracy. However, with the Grid Decomposition Property we are also able to obtain optimal rates for (13).

## III.  Error Analysis

The starting point for an analysis of least squares methods is typically the observation that the solution $\{\phi_\delta, \underline{u}_h\}$ of the discrete problem is a best approximation to $\{\phi_0, \underline{u}_0\}$ in a suitable norm. In our context this norm arises from the bilinear form

(1) $$B((\phi,\underline{u}),(\psi,\underline{v})) = \int_\Omega (\nabla\phi-\underline{u}) \cdot (\nabla\psi-\underline{v}) + (\operatorname{div}(\underline{u})+q\phi)(\operatorname{div}(\underline{v})+q\psi)$$

and is given by

$$(2) \qquad ||| (\phi,\underline{u}) ||| \equiv B((\phi,\underline{u}),(\phi,\underline{u}))^{\frac{1}{2}} .$$

Letting $\epsilon,\underline{e}$ denote the errors [(12) and (13), section 2] we observe that [(8) and (9), section 2] implies the error $\{\epsilon,\underline{e}\}$ is orthogonal to $S_0^\delta \times \underline{V}_0^h$ in the form $B(\cdot,\cdot)$; i.e.,

$$(3) \qquad B((\epsilon,\underline{e}),(\psi^\delta,\underline{v}^h)) = 0 \qquad \text{all} \quad (\psi^\delta,\underline{v}^h) \in S_0^\delta \times \underline{V}_0^h .$$

It follows that $(\psi_h,\underline{u}_h)$ is a best approximation to $(\psi_0,\underline{u}_0)$ in $|||\cdot|||$. That is, we have the following result

Lemma 1.   For all $(\psi^\delta,\underline{v}^h) \in S_0^\delta \times \underline{V}_0^h$

$$(4) \qquad ||| (\epsilon,\underline{e}) ||| \leq ||| (\phi_0 - \psi^\delta, \underline{u}_0 - \underline{v}^h) ||| $$

Observe that (4) and the approximation property in section 2 can be combined to give an error estimate in the norm $||| \cdot |||$. Indeed, it follows immediately from (1) and the fact that $q$ is a bounded function that $B(\cdot,\cdot)$ is a bounded form on $H^1(\Omega) \times \underline{H}^1(\Omega)$, and without a loss of generality we may assume

$$(5) \qquad |B((\phi,\underline{u}),(\psi,\underline{v}))| \leq (|| \phi ||_1 + || \underline{u} ||_1)(|| \psi ||_1 + || \underline{v} ||_1)$$

Thus

$$(6) \qquad ||| (\phi,\underline{u}) ||| \leq || \phi ||_1 + || \underline{u} ||_1 \quad ,$$

and so

$$(7) \qquad ||| (\epsilon,\underline{e}) ||| \leq C_A(\delta^{\ell-1} || \phi_0 ||_\ell + h^{k-1} || \nabla\phi_0 ||_k) \quad .$$

This error estimate is not very important by itself since the reverse of (6) is not valid, i.e., $\||\cdot\||$ is majorized by the norm $\|\cdot\|_1$ on $H^1(\Omega)$ but is not equivalent to it. In fact, it is by no means clear that (2) even dominates the $L_2$ norm $\|\cdot\|_0$.

To obtain error estimates in more reasonable norms, we shall need to exploit the solvability of the boundary value problem [(1)-(3), section 2]. More precisely, we shall need an a priori inequality of the form

$$(8) \qquad \|\psi\|_{2+t} < C_E \|\Delta\psi + q\psi\|_t \qquad t = 0,1$$

to hold for all $\psi \in H^{2+t}(\Omega)$ satisfying the boundary conditions

$$(9) \qquad \psi = 0 \text{ on } \Gamma_0 \quad , \quad \nabla\psi \cdot \underline{\nu} = 0 \text{ on } \Gamma_N$$

(i.e., $(\psi, \nabla\psi) \in S_0 \times \underline{V}_0$). This will be the case for a fixed positive number $C_E$ provided $\Omega$ and $q$ are sufficiently smooth [2].

This regularity property will enable us to establish optimal error estimates for

$$\|\varepsilon\|_0 \quad , \quad \|\text{div } \underline{e}\|_{-1} \quad .$$

To obtain an estimate for

$$\|\underline{e}\|_0 \qquad ,$$

we shall need the Grid Decomposition Property discussed in section 2.

**Lemma 2.**    If  $C_A$ ,  $C_E$   denote the constants in  [(10)-(11), section 2]  and (8), then

(10)          $\| \operatorname{div} \underline{e} + q \|_{-1} \leq C_A C_E (h+\delta) \||(\varepsilon, \underline{e})\||$

**Proof.**      We recall  [2] that

(11)          $\| \operatorname{div}(\underline{e})+q\varepsilon\|_{-1} \leq \sup_{\eta}\{\int_{\Omega} \eta[\operatorname{div}\underline{e}+q\varepsilon] \,\big|\, \|\eta\|_1 \leq 1\}$

Now for any  $\eta \in H^1(\Omega)$   with

$$\|\eta\|_1 \leq 1$$

we solve

$$\Delta\xi + q\xi = \eta \quad \text{in} \quad \Omega$$

(12)

$$\xi = 0 \quad \text{on} \quad \Gamma_D, \quad \nabla\xi \cdot \underline{\nu} = 0 \quad \text{on} \quad \Gamma_N,$$

or what is the same,

$$\underline{p} = \nabla\xi \qquad\qquad\qquad \text{on} \quad \Omega$$

(13)        $\operatorname{div}(\underline{p}) + q\xi = \eta \qquad\qquad \text{on} \quad \Omega$

$$\xi = 0 \quad \text{on} \quad \Gamma_D, \quad \underline{p} \cdot \underline{\nu} = 0 \qquad \text{on} \quad \Gamma_N.$$

Regularity gives

(14)          $\|p\|_2 \leq \|\xi\|_3 \leq C_E \|\eta\|_1 \leq C_E$

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Observe since $p - \underline{\nabla}\xi = \underline{0}$,

$$\int_\Omega \eta[\text{div}(\underline{e}) + q\varepsilon] = \int_\Omega [\text{div } \underline{p} + q\xi][\text{div } \underline{e} + q\varepsilon]$$

$$= B((\varepsilon,\underline{e}),(\xi,\underline{p}))$$

Use of the orthogonality property (3) and (6) leads to

$$(15) \qquad |\int_\Omega \eta[\text{div}(\underline{e}) + q\,\varepsilon]| = |B((\varepsilon,\underline{e}),(\xi-\hat{\xi}_\delta, \underline{p}-\hat{\underline{p}}_h))|$$

$$\leq |||(\varepsilon,\underline{e})||| \quad |||(\xi-\hat{\xi}_\delta,\underline{p}-\hat{\underline{p}}_h)|||$$

$$\leq |||(\varepsilon,\underline{e})||| \; (\|\xi-\hat{\xi}_\delta\|_1 + \|\underline{p}-\hat{\underline{p}}_h\|_1) \;.$$

This inequality is valid for any $\hat{\xi} \in S_0^\delta$ and $\hat{p}_h \in \underline{V}_0^h$, and in particular we choose the latter so that

$$\|\xi-\hat{\xi}_\delta\|_1 \leq C_A \delta \| \;\xi\; \|_2 \leq C_A C_E \delta$$

$$\|\underline{p}-\hat{\underline{p}}_h\|_1 \leq C_A h \|\underline{p}\|_2 \leq C_A C_E h$$

Our estimate (10) now follows by taking the sup in (15) with $\| \eta \|_1 \leq 1$.

Lemma 3.    In the same context as Lemma 2 we have

$$(16) \qquad \| \varepsilon \|_0 \leq C_A C_E \delta \; |||(\varepsilon,\underline{e})||| + C_E(\| q\|_{L^\infty} + 1) \; \| \text{div}(\underline{e}) + q\varepsilon \|_{-1}$$

<u>Proof</u>.        We solve the boundary value problem

(17)                    $\Delta \eta + q \eta = \varepsilon$  in  $\Omega$

                        $\eta = 0$  on  $\Gamma_D$,  $\nabla \eta \cdot \underline{\nu} = 0$  on  $\Gamma_N$.

for  $\eta$.  Regularity gives

(18)                    $\| \eta \|_2 \leq C_E \| \varepsilon \|_0$  .

In addition,

(19)                    $\| \varepsilon \|_0^2 = \int_\Omega \varepsilon(\Delta \eta + q \eta)$  .

Integrating by parts we have

$$\int_\Omega \varepsilon \Delta \eta = \int_\Gamma \varepsilon \nabla \eta \cdot \underline{\nu} - \int_\Omega \nabla \varepsilon \cdot \nabla \eta \quad ,$$

which because of the boundary conditions reduces (19) to

$$\| \varepsilon \|_0^2 = \int_\Omega \{ -\nabla \varepsilon \cdot \nabla \eta + q \varepsilon \eta \} \ .$$

But observe that

$$B((\varepsilon,\underline{e}),(\eta,\underline{0})) = \int_\Omega \{ (\nabla \varepsilon - \underline{e}) \cdot \nabla \eta + (\mathrm{div}(\underline{e}) + q \varepsilon) q \eta \} \ .$$

Integrating by parts once more we have

$$\int_\Omega \underline{e} \cdot \nabla \eta = \int_\Gamma \underline{e} \cdot \underline{\nu} \eta - \int_\Omega \mathrm{div}(\underline{e}) \eta = - \int_\Omega \mathrm{div}(\underline{e}) \eta \ .$$

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The last three equalities combine to give

$$\| \varepsilon \|_0^2 = \int_\Omega (q+1)(\operatorname{div} \underline{e} + q\varepsilon)\eta - B((\varepsilon,\underline{e}),(\eta,\underline{0})) \quad,$$

which with the orthogonality property (3) becomes

$$(20) \qquad \| \varepsilon \|_0^2 = \int_\Omega (q+1)(\operatorname{div} \underline{e} + q\varepsilon)\eta - B((\varepsilon,\underline{e}),(\eta-\hat{\eta}_\delta,\underline{0})) \quad.$$

The function $\hat{\eta}_\delta \in S_0^\delta$ can be chosen so that

$$\| \eta-\hat{\eta}_\delta \|_1 \le C_A \delta \| \eta \|_2 \le C_A C_E \delta \| \varepsilon \|_0 \quad.$$

Since

$$\left| \int_\Omega (q+1)(\operatorname{div}\underline{e} + q\varepsilon)\eta \right| \le (\| q \|_{L^\infty} + 1) \| \operatorname{div}(\underline{e}) + q\varepsilon \|_{-1} \| \eta \|_1$$

$$\le C_E(\| q \|_{L^\infty} + 1) \| \operatorname{div}\underline{e} + q\varepsilon \|_{-1} \| \varepsilon \|_0 \quad,$$

the estimate (16) follows from (20).

**Theorem 1.** There is a constant $C$ depending only on $C_A, C_E$, and $\| q \|_{L^\infty}$ such that

$$(21) \qquad \| \varepsilon \|_0 \le C(h+\delta) ||| (\varepsilon,\underline{e}) |||$$

and

$$(22) \qquad \| \operatorname{div}\underline{e} \|_{-1} \le C(h+\delta) ||| (\varepsilon,\underline{e}) |||$$

In particular,

$$\| \varepsilon \|_0 \leq C\{(h^{k-1}\delta + h^k)\| \nabla\phi_0\|_k + (\delta^{\ell-1}h + \delta^\ell)\| \phi_0\|_\ell \} \quad ,$$

with a similar bound for $\| \text{div}(\underline{e})\|_{-1}$ .


Proof. The inequality (21) is a direct consequence of (10) and (16). The inequality (22) follows from (10) and

$$\| \text{div } \underline{e} \|_{-1} \leq \| \text{div}(\underline{e}) + q\varepsilon \|_{-1} + \| q\varepsilon \|_{-1}$$

with the observation that

$$\| q\varepsilon \|_{-1} \leq \| q\varepsilon \|_0 \leq \| q \|_{L_\infty} \| \varepsilon \|_0 \quad .$$

We now turn to the $L_2$ estimate for $\underline{e}$. The key is the Grid Decomposition Property and we use this to write

$$(23) \qquad \underline{u}_h - \hat{\underline{u}}_h = \underline{w}_h + \underline{z}_h \quad ,$$

where

$$(24) \qquad \text{div}(\underline{z}_h) = 0 \quad , \qquad \int_\Omega \underline{w}_h \cdot \underline{z}_h = 0 \quad ,$$

and

$$(25) \qquad \| \underline{w}_h \|_0 \leq C_S \| \text{div}(\underline{u}_h - \hat{\underline{u}}_h)\|_{-1}$$

Recall that for any $\underline{v} \in \underline{V}_0$

$$\| \text{div } \underline{v} \|_{-1} \leq \| \underline{v} \|_0 \quad .$$

Thus

$$\| \underline{w}_h \|_0 \leq C_S \{ \| \operatorname{div}(\underline{u} - \hat{\underline{u}}_h) \|_{-1} + \| \operatorname{div}(\underline{u}_0 - \underline{u}_h) \|_{-1} \}$$

$$\leq C_S \{ \| \underline{u} - \hat{\underline{u}}_h \|_0 + \| \operatorname{div} \underline{e} \|_1 \} \ .$$

To complete the estimate we must get a similar bound on $\underline{z}_h$. To do this we turn to the identity (3), which can be rewritten with $\psi^\delta = 0$ as

$$\int_\Omega \{ \underline{u}_h \cdot \underline{v}^h + \operatorname{div}(\underline{u}_h) \operatorname{div}(\underline{v}^h) \} + \int_\Omega (q+1) \operatorname{div}(\underline{v}^h) \phi_\delta$$

$$= \int_\Omega \{ \underline{u} \cdot \underline{v}^h + \operatorname{div}(\underline{u}) \operatorname{div}(\underline{v}^h) \} + \int_\Omega (q+1) \operatorname{div}(\underline{v}^h) \phi \ .$$

Putting $\underline{v}^h = \underline{z}_h$ and noting that $\operatorname{div}(\underline{z}_h) = 0$, this becomes

$$\int_\Omega \underline{u}_h \cdot \underline{z}_h = \int_\Omega \underline{u} \cdot \underline{z}_h \ .$$

Thus as $\underline{z}_h$ and $\underline{w}_h$ are orthogonal

$$\int_\Omega \underline{z}_h \cdot \underline{z}_h = \int_\Omega (\underline{u}_h - \hat{\underline{u}}_h) \, \underline{z}_h = \int_\Omega (\underline{u} - \underline{u}_h) \cdot \underline{z}_h \ .$$

It follows that

(26) $$\| \underline{z}_h \|_0 \leq \| \underline{u} - \hat{\underline{u}}_h \|_0 \ .$$

Combining (23), (25), and (26) we obtain the following theorem.

<u>Theorem 2</u>.  If the Grid Decomposition Property holds, then

$$\| \underline{e} \|_0 \leq 2 \| \underline{u} - \hat{\underline{u}}_h \|_0 + C_S ( \| \operatorname{div} \underline{e} \|_{-1} + \| \operatorname{div}(u - \hat{u}_h) \|_{-1} ) \ .$$

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Thus

$$(27) \qquad \| \underline{e} \|_0 \; \leqq \; C\{(h^k + h^{k-1}\delta) \; \| \nabla\phi_0 \|_k + (\delta^{\ell-1}h + \delta^\ell) \| \phi_0 \|_\ell ) \quad .$$

Note (27) implies that the error will be of order $O(h^k)$ if $k = \ell$ and $\delta = O(h)$. That is, the same order polynomials should be used for both $\phi$ and $\underline{u}$, and the associated mesh spacing should be comparable. In practice this can be done by taking the same grids for both.

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